

# Ultimate generalization of Noether's theorem in the realm of Hamiltonian point dynamics

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**Abstract.** Noether's theorem in the realm of point dynamics establishes the correlation of a constant of motion of a Hamilton-Lagrange system with a particular symmetry transformation that preserves the form of the action functional. Although usually derived in the Lagrangian formalism [1, 2], the natural context for deriving Noether's theorem for first-order Lagrangian systems is the Hamiltonian formalism. The reason is that the class of transformations that leave the action functional invariant coincides with the class of canonical transformations. As a result, any invariant of a Hamiltonian system can be correlated with a symmetry transformation simply by means of the canonical transformation rules. As this holds for any invariant, we thereby obtain the most general representation of Noether's theorem. In order to allow for symmetry mappings that include a transformation of time, we must refer to the extended Hamiltonian formalism. This formalism enables us to define generating functions of canonical transformations that also map time and energy in addition to the conventional mappings of canonical space and momentum variables.

As an example for the generalized Noether theorem, a manifest representation of the symmetry transformation is derived that corresponds to the Runge-Lenz invariant of the Kepler system.

## 1. Introduction

Even more than hundred years after the emerging of Einstein's special theory of relativity, the presentation of classical dynamics in terms of the Lagrangian and the Hamiltonian formalism is still usually based in literature on the Newtonian absolute time as the system evolution parameter. The idea how the Hamilton-Lagrange formalism is to be generalized in order to be compatible with special relativity is obvious and well-established. It consists of introducing a system evolution parameter,  $s$ , as the new independent variable, and of subsequently treating the time  $t = t(s)$  as a *dependent* variable of  $s$ , in parallel to all configuration space variables  $q^i(s)$ .

In order to preserve the canonical form of the the action functional, we must introduce an *extended* Hamiltonian,  $H_e$ . Setting up the correlation of extended and conventional Hamiltonians, the *crucial point* is that we must not confuse the conventional Hamilton *function*,  $H$ , with its *value*,  $e$ . Its negative,  $-e(s)$ , then plays the role of the *additional canonical variable* that is conjugate to the time variable,  $t(s)$ .

With our relation of  $H_e$  and  $H$  in place, we find the subsequent extended set of canonical equations to perfectly coincide in its *form* with the conventional one, which means that no

additional functions are involved. This is also true for the theory of extended canonical transformations.

It will be shown that the most general form of Noether's theorem for Hamiltonian point dynamics can be represented by a one-parameter *infinitesimal canonical transformation*. Namely, the characteristic function of the generator of an infinitesimal transformation must be a constant of motion in order for the subsequent transformation to be *canonical*, hence to preserve the action functional. Then the canonical transformation rules embody the symmetry relations of the dynamical system that correspond to this constant of motion.

As a non-trivial example of the correlation of a system's invariant with a symmetry transformation that leaves the action functional invariant, we present a particular symmetry for the Runge-Lenz invariant of the classical Kepler system that is associated with a non-zero time shift.

## 2. Generalized action functional

The state of a classical dynamical system of  $n$  degrees of freedom at time  $t$  is completely described by  $\mathbf{q} = (q^1, \dots, q^n)$  the vector of generalized space coordinates and  $\mathbf{p} = (p_1, \dots, p_n)$  the covector of generalized momenta. We assume the system to be described by a Hamiltonian

$$H : \mathbb{R}^{2n} \times \mathbb{R} \rightarrow \mathbb{R}, \quad e = H(\mathbf{q}, \mathbf{p}, t). \quad (1)$$

A Hamiltonian  $H$  contains the *complete information* on the given dynamical system through the dependence of its *value*  $e$  on each  $q^i$  and each  $p_i$  along the time axis  $t$ .

In order to formulate the principle of least action — originating from Leibniz, Maupertuis, Euler, and Lagrange — we define the action functional  $\Phi(\gamma)$  as the line integral

$$\Phi(\gamma) = \int_{\gamma} \sum_{i=1}^n p_i dq^i - H(\mathbf{q}, \mathbf{p}, t) dt, \quad (2)$$

hence as a mapping of the set of phase-space paths  $\gamma \subset \mathbb{R}^{2n} \times \mathbb{R}$  into  $\mathbb{R}$ . A phase-space path is defined as the smooth mapping that connects a system's *initial state*  $(\mathbf{q}_0, \mathbf{p}_0, t_0)$  with a fixed *final state*  $(\mathbf{q}_1, \mathbf{p}_1, t_1)$ ,

$$\gamma : \{(\mathbf{q}, \mathbf{p}, t) \in \mathbb{R}^{2n+1} \mid (\mathbf{q}_0, \mathbf{p}_0, t_0) \mapsto (\mathbf{q}_1, \mathbf{p}_1, t_1)\}.$$

The phase-space path  $\gamma_{\text{ext}}$  the dynamical system actually realizes follows from the *principle of least action*. It states that the variation of the action functional (2) vanishes for  $\gamma_{\text{ext}}$ , hence  $\delta\Phi(\gamma_{\text{ext}}) = 0$ . Commonly, a restricted path  $\gamma_{\text{r}} \subset \mathbb{R}^{2n}$  is defined by parameterizing (2) in terms of the system's *time*,  $t$

$$\gamma_{\text{r}} : \{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n} \mid q^i = q^i(t), p_i = p_i(t); t_0 \leq t \leq t_1\}.$$

The line integral (2) is thus converted into a conventional integral. The action principle then writes

$$\delta\Phi(\gamma) = \delta \int_{t_0}^{t_1} \left[ \sum_{i=1}^n p_i(t) \frac{dq^i(t)}{dt} - H(\mathbf{q}(t), \mathbf{p}(t), t) \right] dt \stackrel{!}{=} 0. \quad (3)$$

From the calculus of variations, one finds that the functional  $\Phi(\gamma_{\text{ext}})$  takes on an *extreme* ( $\delta\Phi(\gamma_{\text{ext}}) = 0$ ), exactly if the phase-space path  $(\mathbf{q}(t), \mathbf{p}(t))$  satisfies the “canonical equations” ( $i = 1, \dots, n$ ),

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{de}{dt} = \frac{\partial H}{\partial t}, \quad (4)$$

where  $e(t)$  denotes according to Eq. (1) the *instantaneous value* of the Hamiltonian  $H$ .

If the Hamiltonian  $H$  depends explicitly on time  $t$ , then the parametrization of the line integral (2) in terms of  $t$  as in Eq. (3) is of restricted utility. The most general parametrization of the variational problem  $\delta\Phi(\gamma) \stackrel{!}{=} 0$  is encountered if we treat the time  $t = t(s)$  as a canonical variable and parameterize the line integral in terms of a system evolution parameter,  $s$  [3]

$$\delta \int_{s_0}^{s_1} \left[ \sum_{i=1}^n p_i(s) \frac{dq^i(s)}{ds} - H(\mathbf{q}(s), \mathbf{p}(s), t(s)) \frac{dt(s)}{ds} \right] ds \stackrel{!}{=} 0.$$

The  $q^i$  and time  $t$  are now treated on *equal footing*. The symmetric form of the integrand suggests to define the  $2n + 2$  dimensional *extended phase space* by introducing

$$q^0(s) \equiv ct(s), \quad p_0(s) \equiv -e(s)/c$$

as an *additional pair* of canonically conjugate coordinates. Herein,  $e = e(s) \in \mathbb{R}$  is the *instantaneous value* of the Hamiltonian  $H(\mathbf{q}, \mathbf{p}, t)$  at  $s$ , but *not* the *function*  $H$ . In contrast to  $H$ , the canonical coordinate  $p_0 = -e/c$  constitutes a function of the independent variable  $s$  only, hence exhibits no derivative other than that with respect to  $s$ ,

$$e(s) \not\equiv H(\mathbf{q}(s), \mathbf{p}(s), t(s)).$$

The  $s$ -parametrized action functional can be converted into the standard form of Eq. (3)

$$\delta \int_{s_0}^{s_1} \left[ \sum_{\alpha=0}^n p_\alpha(s) \frac{dq^\alpha(s)}{ds} - H_e(\mathbf{q}(s), \mathbf{p}(s), t(s), e(s)) \right] ds \stackrel{!}{=} 0 \quad (5)$$

if we define the *extended* Hamiltonian  $H_e$  as [4, 5]

$$H_e(\mathbf{q}, \mathbf{p}, t, e) \equiv \left[ H(\mathbf{q}, \mathbf{p}, t) - e \right] \frac{dt}{ds}. \quad (6)$$

With  $H_e$ , we encounter the *extended* functional (5) exactly in the form of the *conventional* functional (3). Note that the sum in (5) now includes terms related to  $q^0 = ct$  and  $p_0 = -e/c$ . As  $H(\mathbf{q}, \mathbf{p}, t) \stackrel{!}{=} e$ , the extended Hamiltonian  $H_e$  actually represents an *implicit function*,

$$H_e(\mathbf{q}, \mathbf{p}, t, e) \stackrel{!}{=} 0. \quad (7)$$

Of course, this does *not* mean that  $H_e$  may be eliminated from the action functional (5) since the partial derivatives of  $H_e$  do not vanish and hence enter into the calculation of the variation.

The action functional (2) can now be written equivalently in terms of the extended Hamiltonian  $H_e$  as

$$\Phi(\gamma) = \int_{\gamma} \sum_{\alpha=0}^n p_\alpha dq^\alpha - H_e(\mathbf{q}, \mathbf{p}, t, e) ds,$$

with the paths  $\gamma$  being defined as the set of smooth mappings

$$\gamma : \{ (\mathbf{q}, \mathbf{p}, t, e) \in \mathbb{R}^{2n+1} \mid (\mathbf{q}_0, \mathbf{p}_0, t_0, e_0) \mapsto (\mathbf{q}_1, \mathbf{p}_1, t_1, e_1); H_e = 0 \}.$$

Similar to the case of Eq. (3) but now taking  $s$  as the system's parameter, the variation of the generalized functional (5) vanishes if the *non-restricted* phase-space path  $\gamma \subset \mathbb{R}^{2n+1}$

$$\gamma : \{(\mathbf{q}, \mathbf{p}, t, e) \in \mathbb{R}^{2n+1} \mid q^i = q^i(s), p_i = p_i(s), t = t(s); e = e(s), H_e = 0; s_0 \leq s \leq s_1\}$$

satisfies the *extended* set of canonical equations

$$\frac{dq^i}{ds} = \frac{\partial H_e}{\partial p_i}, \quad \frac{dp_i}{ds} = -\frac{\partial H_e}{\partial q^i}, \quad \frac{dt}{ds} = -\frac{\partial H_e}{\partial e}, \quad \frac{de}{ds} = \frac{\partial H_e}{\partial t}. \quad (8)$$

The number of canonical equations is now *even*. We have thus converted the *pre-symplectic* conventional Hamiltonian formalism into an extended *symplectic* description. Geometrically, the system's motion now takes place on a *hyper-surface*, defined by  $H_e = 0$ , within the cotangent bundle  $T^*(\mathbb{M} \times \mathbb{R})$  over the space-time configuration manifold  $\mathbb{M} \times \mathbb{R}$ . This contrasts with the conventional, unconstrained Hamiltonian description on the pre-symplectic cotangent bundle  $(T^*\mathbb{M}) \times \mathbb{R}$ . On the other hand, the extended description parallels that of a conventional Hamiltonian with no *explicit* time dependence,  $H(\mathbf{q}, \mathbf{p}) = e_0$ , where the system's initial energy  $e_0$  embodies a *constant of motion*. In that case, the system's motion again takes place on a *hyper-surface* that is now defined by  $H(\mathbf{q}, \mathbf{p}) = e_0$  within the cotangent bundle  $T^*\mathbb{M}$  over the configuration manifold  $\mathbb{M}$ .

### 3. Example: Relativistic particle in an external potential $V$

An example for a *non-trivial* extended Hamiltonian is furnished by  $H_e$  of a relativistic particle in an external potential,

$$H_e(\mathbf{q}, \mathbf{p}, t, e) = \frac{1}{2m} \left[ \mathbf{p}^2 - \left( \frac{e - V(\mathbf{q}, t)}{c} \right)^2 \right] + \frac{1}{2}mc^2. \quad (9)$$

Due to the constraint  $H_e = 0$  from Eq. (7), we can solve Eq. (9) for  $e$  to find the equivalent *conventional* Hamiltonian  $H$  as the right-hand side of the equation  $e = H$ ,

$$e = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} + V(\mathbf{q}, t) = H(\mathbf{q}, \mathbf{p}, t). \quad (10)$$

The conventional Hamiltonian  $H$  corresponding to  $H_e$  from Eq. (9) is no longer a *quadratic form* in the canonical momenta. To derive the corresponding quantum equation, the *canonical quantization rules*

$$p_\mu \mapsto \hat{p}_\mu = -i\hbar \frac{\partial}{\partial q^\mu}, \quad e \mapsto \hat{e} = i\hbar \frac{\partial}{\partial t}, \quad q^\mu \mapsto \hat{q}^\mu = q^\mu \mathbb{1}, \quad t \mapsto \hat{t} = t \mathbb{1}$$

may thus be applied to the extended Hamiltonian  $H_e = 0$  only, which here yield the Klein-Gordon equation.

The canonical equation for  $dt/ds$  is obtained as

$$\frac{dt}{ds} = -\frac{\partial H_e}{\partial e} = \frac{e - V}{mc^2} = \frac{\sqrt{\mathbf{p}^2 c^2 + m^2 c^4}}{mc^2} = \sqrt{1 + \left( \frac{\mathbf{p}}{mc} \right)^2} = \gamma.$$

Thus, if  $t$  quantifies the laboratory time, then  $s$  measures the particle's *proper time*. We easily convince ourselves that the other three canonical equations emerging from  $H_e$  according to Eqs. (8) coincide with the conventional canonical equations emerging from  $H$  according to Eqs. (4). Thus,  $H_e$  from Eq. (9) and  $H$  from Eq. (10) indeed describe *the same physical system*.

#### 4. Extended canonical transformations

As usual, the general condition for a transformation to be *canonical* is to preserve the *form* of the action functional. In the extended description, where time  $q^0 \equiv ct$  and  $p_0 \equiv -e/c$  are canonical conjugate dynamical variables, this means that now the form of the *extended* action principle from Eq. (5) must be preserved, hence

$$\delta \int_{s_1}^{s_2} \left[ \sum_{\alpha=0}^n p_\alpha \frac{dq^\alpha}{ds} - H_e \right] ds = \delta \int_{s_1}^{s_2} \left[ \sum_{\alpha=0}^n P_\alpha \frac{dQ^\alpha}{ds} - H'_e \right] ds.$$

For this requirement to hold, the *integrands* may differ at most by the total derivative  $dF_1/ds$  of a function  $F_1(\mathbf{q}, \mathbf{Q}, t, T)$ , with  $q^0 \equiv ct, Q^0 \equiv cT$ . Comparing the coefficients of the derivatives in the action functionals with

$$\frac{dF_1}{ds} = \sum_{\alpha=0}^n \left( \frac{\partial F_1}{\partial q^\alpha} \frac{dq^\alpha}{ds} + \frac{\partial F_1}{\partial Q^\alpha} \frac{dQ^\alpha}{ds} \right),$$

we find the canonical transformation rules for an extended generating function of type  $F_1(\mathbf{q}, \mathbf{Q}, t, T)$ ,

$$p_i = \frac{\partial F_1}{\partial q^i}, \quad P_i = -\frac{\partial F_1}{\partial Q^i}, \quad e = -\frac{\partial F_1}{\partial t}, \quad E = \frac{\partial F_1}{\partial T}, \quad H'_e = H_e.$$

The value of the extended Hamiltonian  $H_e$  is thus conserved under extended canonical transformations, which means that the physical motion is kept being confined to the surface  $H'_e = 0$ . Of course, the functional dependence on the respective set of canonical variables will be different for  $H_e$  and  $H'_e$ , in general.

The Legendre transformation

$$F_2(\mathbf{q}, \mathbf{P}, t, E) = F_1(\mathbf{q}, \mathbf{Q}, t, T) + \sum_{i=1}^n Q^i P_i - TE$$

yields an *equivalent*, more useful set of canonical transformation rules

$$p_i = \frac{\partial F_2}{\partial q^i}, \quad Q^i = \frac{\partial F_2}{\partial P_i}, \quad e = -\frac{\partial F_2}{\partial t}, \quad T = \frac{\partial F_2}{\partial E}, \quad H'_e = H_e. \quad (11)$$

According to Eq. (6), the transformation rule  $H'_e = H_e$  for the extended Hamiltonians can be expressed in terms of conventional Hamiltonians as

$$\left[ H'(\mathbf{Q}, \mathbf{P}, T) - E \right] \frac{dT}{ds} = \left[ H(\mathbf{q}, \mathbf{p}, t) - e \right] \frac{dt}{ds}.$$

Eliminating the evolution parameter  $s$ , we arrive at the following two equivalent transformation rules for the conventional Hamiltonians under extended canonical transformations

$$\begin{aligned} \left[ H'(\mathbf{Q}, \mathbf{P}, T) - E \right] \frac{\partial T}{\partial t} &= H(\mathbf{q}, \mathbf{p}, t) - e \\ \left[ H(\mathbf{q}, \mathbf{p}, t) - e \right] \frac{\partial t}{\partial T} &= H'(\mathbf{Q}, \mathbf{P}, T) - E. \end{aligned} \quad (12)$$

The transformation rules are generalizations of the rule for conventional canonical transformations as cases with  $T \neq t$  are now included.

#### 4.1. Extended generating function of a conventional canonical transformation

An important example of an extended generating function is the particular  $F_2$  that defines a *conventional* canonical transformation. Consider the particular extended generating function

$$F_2(\mathbf{q}, \mathbf{P}, t, E) = f_2(\mathbf{q}, \mathbf{P}, t) - tE, \quad (13)$$

with  $f_2(\mathbf{q}, \mathbf{P}, t)$  denoting a conventional generating function. The coordinate transformation rules (11) for this  $F_2$  follow as

$$p_i = \frac{\partial f_2}{\partial q^i}, \quad Q^i = \frac{\partial f_2}{\partial P_i}, \quad e = -\frac{\partial f_2}{\partial t} + E, \quad T = t.$$

Since  $\partial T / \partial t = 1$ , the extended transformation rule for conventional Hamiltonians from Eq. (12) simplifies to

$$H' - E = H - e \quad \implies \quad H'(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}, \mathbf{p}, t) + \frac{\partial f_2}{\partial t}.$$

The partial derivatives of  $f_2$  obviously yield the usual conventional canonical transformation rules. The particular extended generating function  $F_2$  from Eq. (13) thus defines the conventional canonical transformation generated by  $f_2$ . We conclude that the group of conventional canonical transformations establishes a *subgroup* of the group of extended canonical transformations.

### 5. Generalized Noether theorem

We are now prepared to derive the generalized Noether theorem in the Hamiltonian formalism on the basis of an extended infinitesimal canonical transformation. The extended generating function of an *infinitesimal* canonical transformation is

$$F_2(\mathbf{q}, \mathbf{P}, t, E) = \sum_{i=1}^n q^i P_i - tE + \delta\epsilon I(\mathbf{q}, \mathbf{p}, t, e), \quad (14)$$

with  $\delta\epsilon \neq 0$  a small parameter and  $I(\mathbf{q}, \mathbf{p}, t, e)$  a function of the set of extended phase-space variables. The subsequent transformation rules (11) are

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q^i} = P_i + \delta\epsilon \frac{\partial I}{\partial q^i}, & e &= -\frac{\partial F_2}{\partial t} = E - \delta\epsilon \frac{\partial I}{\partial t} \\ Q^i &= \frac{\partial F_2}{\partial P_i} = q^i + \delta\epsilon \frac{\partial I}{\partial P_i}, & T &= -\frac{\partial F_2}{\partial E} = t - \delta\epsilon \frac{\partial I}{\partial e} \\ H'_e &= H_e. \end{aligned}$$

To *first order* in  $\delta\epsilon$ , the variations  $\delta p_i$ ,  $\delta q^i$ ,  $\delta e$ ,  $\delta t$ , and  $\delta H_e$  follow as

$$\begin{aligned} \delta p_i &\equiv P_i - p_i = -\delta\epsilon \frac{\partial I}{\partial q^i}, & \delta e &\equiv E - e = \delta\epsilon \frac{\partial I}{\partial t} \\ \delta q^i &\equiv Q^i - q^i = \delta\epsilon \frac{\partial I}{\partial p_i}, & \delta t &\equiv T - t = -\delta\epsilon \frac{\partial I}{\partial e} \\ \delta H_e &\equiv H'_e - H_e = 0. \end{aligned} \quad (15)$$

The transformation rule  $H'_e = H_e$  for the extended Hamiltonian ensures that the constraint  $H_e = 0$  is maintained in the transformed system, hence that  $H'_e = 0$ . On the other hand, the variation of  $H_e$  due to variations of the canonical variables is

$$\delta H_e = \sum_{i=1}^n \left( \frac{\partial H_e}{\partial q^i} \delta q^i + \frac{\partial H_e}{\partial p_i} \delta p_i \right) + \frac{\partial H_e}{\partial t} \delta t + \frac{\partial H_e}{\partial e} \delta e.$$

Inserting the variations from the transformation rules (15), we must make sure that the requirement  $\delta H_e = 0$  actually holds in order for the transformation to be *canonical*,

$$\begin{aligned}\delta H_e &= \delta\epsilon \left[ \sum_{i=1}^n \left( \frac{\partial H_e}{\partial q^i} \frac{\partial I}{\partial p_i} - \frac{\partial H_e}{\partial p_i} \frac{\partial I}{\partial q^i} \right) - \frac{\partial H_e}{\partial t} \frac{\partial I}{\partial e} + \frac{\partial H_e}{\partial e} \frac{\partial I}{\partial t} \right] \\ &= \delta\epsilon [H_e, I]_{\text{ext}} \stackrel{!}{=} 0.\end{aligned}\tag{16}$$

Herein  $[H_e, I]_{\text{ext}}$  defines the extended Poisson bracket. Thus, the requirement  $\delta H_e = 0$  from Eqs. (15) for a transformation to be *canonical* is satisfied if and only if the function  $I(\mathbf{q}, \mathbf{p}, t, e)$  in the generating function “commutes” with the system’s extended Hamiltonian  $H_e$ . Along the system’s phase-space trajectory, the canonical equations (8) apply, hence

$$\begin{aligned}\delta H_e &= \delta\epsilon \left[ \sum_{i=1}^n \left( -\frac{dp_i}{ds} \frac{\partial I}{\partial p_i} - \frac{dq^i}{ds} \frac{\partial I}{\partial q^i} \right) - \frac{de}{ds} \frac{\partial I}{\partial e} - \frac{dt}{ds} \frac{\partial I}{\partial t} \right] \\ &= -\delta\epsilon \frac{dI}{ds} \stackrel{!}{=} 0.\end{aligned}$$

We can now express the generalized Noether theorem and its inverse in the extended Hamiltonian formalism as:

**Theorem 1 (generalized Noether)** *The characteristic function  $I(\mathbf{q}, \mathbf{p}, t, e)$  in the extended generating function  $F_2$  from Eq. (14) must be a constant of motion in order to define a canonical transformation. The subsequent transformation rules (15) then comprise an infinitesimal one-parameter symmetry transformation that preserves the form of the action functional (5).*

*Conversely, if a one-parameter symmetry transformation is known to preserve the form of the extended action functional (5), then the transformation is canonical, and hence can be derived from a generating function. The characteristic function  $I(\mathbf{q}, \mathbf{p}, t, e)$  in the corresponding infinitesimal generating function (14) then represents a constant of motion.*

We may reformulate the generalized Noether theorem in terms of a conventional Hamiltonian  $H$  with the time  $t$  the independent variable. From the correlation (6) of extended and conventional Hamiltonians, one finds

$$\frac{\partial H_e}{\partial t} = \frac{\partial H}{\partial t} \frac{dt}{ds}, \quad \frac{\partial H_e}{\partial e} = -\frac{dt}{ds}, \quad \frac{\partial H_e}{\partial q^i} = \frac{\partial H}{\partial q^i} \frac{dt}{ds}, \quad \frac{\partial H_e}{\partial p_i} = \frac{\partial H}{\partial p_i} \frac{dt}{ds}.$$

In terms of a conventional Hamiltonian  $H$ , the commutation condition from Eq. (16) for  $\delta H_e = 0$  is converted into

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial e} \frac{\partial H}{\partial t} + \sum_{i=1}^n \left( \frac{\partial I}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial I}{\partial p_i} \frac{\partial H}{\partial q^i} \right) = 0.$$

Due to the conventional canonical equations (4) this is equivalent to

$$\frac{d}{dt} I(\mathbf{q}, \mathbf{p}, t, e) = 0.\tag{17}$$

The infinitesimal symmetry transformation rules (15) that are associated with an invariant  $I$  are

$$\delta p_i = -\delta\epsilon \frac{\partial I}{\partial q^i}, \quad \delta q^i = \delta\epsilon \frac{\partial I}{\partial p_i}, \quad \delta e = \delta\epsilon \frac{\partial I}{\partial t}, \quad \delta t = -\delta\epsilon \frac{\partial I}{\partial e}.\tag{18}$$

The condition (17) in conjunction with the one-parameter infinitesimal symmetry transformation (18) comprises the mathematical kernel of the generalized Noether theorem in the realm of point dynamics.

## 6. Examples: Kepler system invariants and their symmetries

### 6.1. Rotational symmetry and angular momentum conservation

The classical Kepler system is a two-body problem with the mutual interaction following an inverse square force law. In the frame of the reference body, the Cartesian coordinates  $q_1, q_2$  of its counterpart may be described in the plane of motion by

$$\ddot{q}_i + \mu(t) \frac{q_i}{\sqrt{(q_1^2 + q_2^2)^3}} = 0, \quad i = 1, 2, \quad (19)$$

with  $\mu(t) = G[m_1(t) + m_2(t)]$  the possibly time-dependent gravitational coupling strength that is induced by possibly time-dependent masses  $m_1(t)$  and  $m_2(t)$  of the interacting bodies. We may regard the equation of motion (19) to originate from the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}, t) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + V(\mathbf{q}, t) \quad (20)$$

containing the interaction potential

$$V(\mathbf{q}, t) = -\frac{\mu(t)}{\sqrt{q_1^2 + q_2^2}} = -\frac{\mu(t)}{r}.$$

As the potential spatially depends on the distance  $r = \sqrt{q_1^2 + q_2^2}$  only, it is obviously invariant with respect to rotations in configuration space,  $(q_1, q_2)$

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad (21)$$

with the parameter  $\epsilon$  denoting the counterclockwise rotation angle. As the transformation depends on the parameter  $\epsilon$  only and not on the canonical coordinates, we refer to it as a *global* symmetry transformation.

This symmetry is maintained if we choose  $\epsilon \equiv \delta\epsilon$  to be very small. Then  $\cos \delta\epsilon \approx 1$ ,  $\sin \delta\epsilon \approx \delta\epsilon$ , and the infinitesimal rules are

$$\delta q_1 \equiv Q_1 - q_1 = \delta\epsilon q_2, \quad \delta q_2 \equiv Q_2 - q_2 = -\delta\epsilon q_1.$$

These rules can be regarded as being derived from the generating function of the *infinitesimal* canonical transformation

$$F_2(q_1, q_2, P_1, P_2, t, E) = -tE + q_1 P_1 + q_2 P_2 + \delta\epsilon (p_1 q_2 - p_2 q_1). \quad (22)$$

According to Noether's theorem, the expression proportional to the parameter  $\delta\epsilon$  must be a constant of motion in order for  $F_2$  to define a canonical transformation, and hence to preserve the physical system. Thus

$$I = p_1 q_2 - p_2 q_1, \quad \frac{dI}{dt} = 0,$$

which establishes the well-known *conservation law of angular momentum* in — possibly time-dependent — central-force fields. With a given symmetry transformation, we have thus determined the corresponding constant of motion.

As with any generating function of a canonical transformation, we can derive from the generating function (22) the rules of both the configuration space coordinates and the respective canonical momenta. In matrix form, the infinitesimal rules for the momenta are

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = [\mathbb{1} + \mathbb{A}_{\delta\epsilon}] \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \mathbb{A}_{\delta\epsilon} = \delta\epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$



with  $\mathbb{1}$  denoting the  $2 \times 2$  unit matrix. The corresponding *finite* transformation is then

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \exp(\mathbb{A}_\epsilon) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \exp(\mathbb{A}_\epsilon) = \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix},$$

which coincides with the transformation of the configuration space variables from Eq. (21). This reflects the fact that the Hamiltonian (20) is equally invariant under rotations in momentum space.

### 6.2. Symmetry transformations associated with the Runge-Lenz invariant

In the Hamiltonian formulation, the converse is also true: if  $I$  denotes a constant of motion of a dynamical system, then the associated infinitesimal symmetry transformation is given by the canonical transformation rules emerging from the one-parameter generating function (14). As an example, a particularly transparent representation of the symmetry transformation that corresponds to the Runge-Lenz invariant of the time-independent case of the Kepler system (20) is derived in the following by admitting a symmetry transformation  $(q_1, q_2)|_t \mapsto (Q_1, Q_2)|_{t+\delta t}$  that is associated with a *time shift*  $\delta t$ .

For constant  $\mu$ , hence constant masses  $m_1$  and  $m_2$  of the interacting bodies, one component of the constant Runge-Lenz vector is expressed in terms of canonicals variables as

$$I(q_1, q_2, p_1, p_2) = -q_1 p_2^2 + q_2 p_1 p_2 + \mu \frac{q_1}{\sqrt{q_1^2 + q_2^2}}. \quad (23)$$

Inserting directly the invariant (23) as the characteristic function  $I$  into the infinitesimal generating function (14), the subsequent canonical transformation rules (18) then define the — rather intricate — corresponding infinitesimal symmetry transformation that preserves the action functional (5). As  $I$  in the form of Eq. (23) does not depend on  $e$ , we have  $\delta t = 0$ , hence *no time shift* is associated with the symmetry transformation.

A more conspicuous representation of the symmetry transformation emerges if we express the invariant  $I$  in extended phase-space variables. With the canonical variable  $e$  being defined as the *value* of  $H$ , hence

$$e = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{\mu}{\sqrt{q_1^2 + q_2^2}},$$

we can replace the square root term in  $I$  from Eq. (23). The invariant then acquires the *equivalent* form

$$I(q_1, q_2, p_1, p_2, e) = \frac{1}{2}q_1 p_1^2 + q_2 p_1 p_2 - \frac{1}{2}q_1 p_2^2 - q_1 e. \quad (24)$$

In contrast to the conventional symmetry analysis (cf, for instance, Ref. [6], p. 121), the invariant  $I$  now depends on the canonical energy variable,  $e$ , which entails a representation of the symmetry transformation with  $\delta t \neq 0$ . Thus, the one-parameter symmetry transformation is now associated with both, a shift of the  $p_i, q_i$  and a shift of time  $t$ . Explicitly, the infinitesimal transformation rules (18) associated with  $I$  from Eq. (24) are

$$\begin{aligned} \delta p_1 &= \delta \epsilon \left( \frac{1}{2}p_2^2 - \frac{1}{2}p_1^2 + e \right), & \delta p_2 &= \delta \epsilon p_1 p_2 \\ \delta q_1 &= \delta \epsilon (q_1 p_1 + q_2 p_2), & \delta q_2 &= \delta \epsilon (p_1 q_2 - p_2 q_1) \\ \delta e &= 0, & \delta t &= \delta \epsilon q_1. \end{aligned}$$

The transformation rules for the new configuration space  $Q_1, Q_2$  variables depend *linearly* on the original ones,  $q_1, q_2$ . We may thus rewrite the infinitesimal configuration space transformation  $Q_i = q_i + \delta q_i$ ,  $i = 1, 2$  in matrix form as

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \Big|_T = [\mathbb{1} + \mathbb{A}_{\delta \epsilon}] \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \Big|_t, \quad \mathbb{A}_{\delta \epsilon} = \delta \epsilon \begin{pmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{pmatrix} \Big|_t,$$

wherein  $\mathbb{1}$  denotes the  $2 \times 2$  unit matrix. With  $\delta\epsilon$  still an *infinitesimal* quantity, this transformation can be written equivalently in terms of the matrix exponential  $\exp(A_{\delta\epsilon})$ ,

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \Big|_{t+\delta t} = e^{\delta\phi} \begin{pmatrix} \cos \delta\psi & \sin \delta\psi \\ -\sin \delta\psi & \cos \delta\psi \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \Big|_t,$$

with  $\delta t = q_1 \delta\epsilon$  and  $\delta\phi = p_1 \delta\epsilon$ ,  $\delta\psi = p_2 \delta\epsilon$ . The system symmetry that corresponds to the Runge-Lenz invariant is thus that the configuration space variables  $Q_i$  at time  $T = t + \delta t$  are correlated with the  $q_i$  at time  $t$  by a *local scaled rotation*. The infinitesimal transformation depends on the actual system coordinates. It is, therefore, referred to as a *local* symmetry transformation.

## 7. Conclusions

Parameterizing the action functional  $\Phi(\gamma)$  in terms of a “system evolution parameter”,  $s$ , enables us to put the space-time variables  $t = t(s)$  and  $q^i = q^i(s)$  on equal footing. The generalization of the Hamiltonian description of dynamics completely retains its canonical form: the conventional set of canonical equations follows from the extended set for the particular extended Hamiltonian  $H_e \equiv H - e = 0$ , whereas conventional canonical transformations simply constitute a subgroup of extended canonical transformations. In the context of the extended Hamiltonian formalism, we can define canonical transformations that additionally map the *time scales* of source and target systems. This sets the stage for deriving the generalized Noether theorem by establishing the connection of a system’s constant of motion with a corresponding canonical transformation that may include a transformation of time. The constant of motion enters as the characteristic function  $I$  into the generating function  $F_2$  of an extended infinitesimal canonical transformation. The subsequent set of canonical transformation rules emerging from this generating function  $F_2$  then establishes the pertaining infinitesimal one-parameter symmetry transformation that preserves the action functional. As this correlation holds for *all* system invariants  $I$ , the distinction between Noether- and non-Noether invariants is obsolete. We thus encounter the *most general form* of Noether’s theorem in the realm of Hamiltonian point dynamics.

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